Maximum cut and eigenvalues

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Workshop - Teoria Espectral de Grafos

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Introduction

What is the Spectral Graph Theory (SGT)?

Research area that aims to obtain network topological information by eigenvalues and eigenvectors of matrices associated to a graph.

Introduction

What is the Spectral Graph Theory (SGT)?

Spectral graph theory is the study of the relationship between a graph and the eigenvalues of matrices naturally associated to that graph.

Introduction

SGT and Combinatorics

The study of eigenvalues of graphs is an important part of combinatorics [for instance, maximum cut, maximum clique, and the chromatic number are related to eigenvalues].

Historically, the first relation between the spectrum and the structure of a graph was discovered in 1876 by Kirchhoff when he proved his famous matrix-tree theorem.

SGT in Brazil

Profa Nair Abreu (COPPE/UFRJ)

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

Let $(i, j) \in E$ denote an edge of G.

Write $w_{ij} \in R$ for the weight of edge $(i, j) \in E$.

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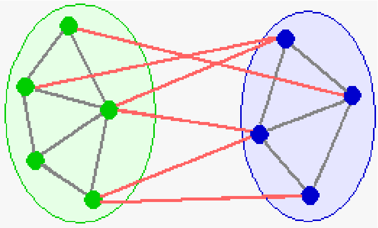
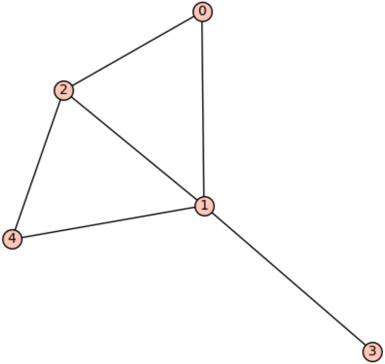
A **cut set** $\delta(S)$ associated with $S \subset V$ is given by the edges

$$\{(i, j) \in E | i \in S,\ j \notin S\}.$$



Consider the graph G below:



Cut set: $\delta(S) = \{(1,3)\}$

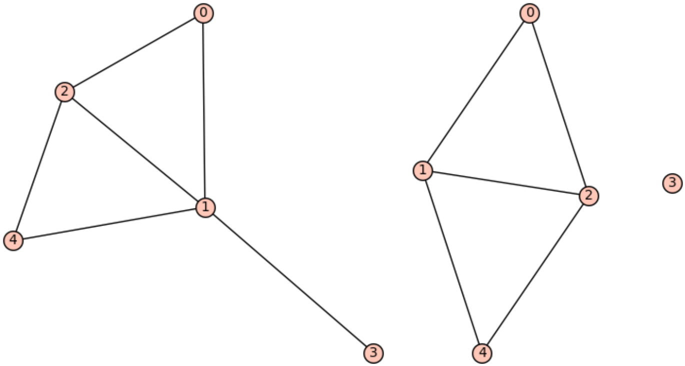


Figure: Graph G and the graph obtained after removing the cut set

Cut set: $\delta(S) = \{(0,1),(1,2),(1,4)\}$

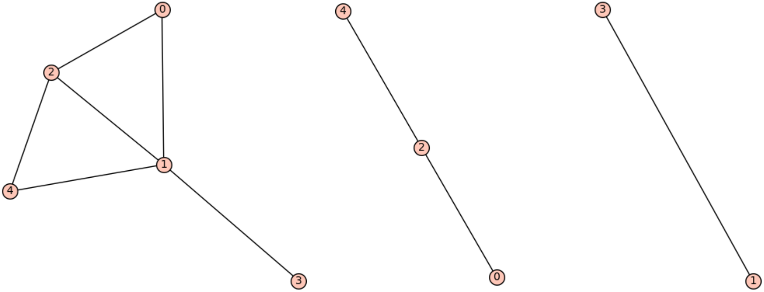


Figure: Graph G and the graph obtained after removing the cut set

The connection between eigenvalues and cuts in graphs has been first discovered by Fiedler (1973).

Some results relating Eigenvalues and Cuts is the subject of this talk.

Before presenting some results of Eigenvalues and Cuts...

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Let us introduce some definitions.

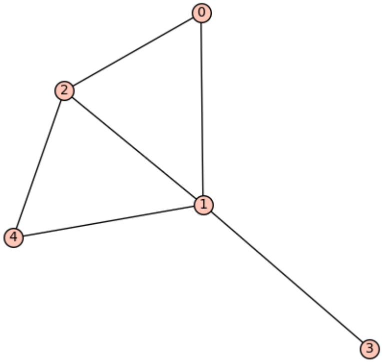
Adjacency matrix of a graph

Let G be a graph on n vertices. The adjacency matrix of G, denoted by $A(G)$, is of order n with entries given by

$$a_{ij} = \begin{cases} 1, & \textit{if } (v_i, v_j) \in E \text{ for } v_i,\ v_j \in\ V; \\ 0, & \text{otherwise .} \end{cases}$$

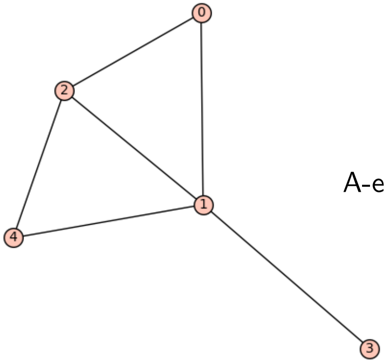
Example

Grafo G



Example

Grafo G



$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

A-eigenvalues: $\{2.68, 0.33, 0, -1.27, -1.74\}$

Laplacian matrix of a graph

Let $D(G)$ be the diagonal matrix of the vertex degrees of G such that $D_{ii} = d(v_i)$ and let $A(G)$ be the adjacency matrix of G. The Laplacian matrix of G is given by

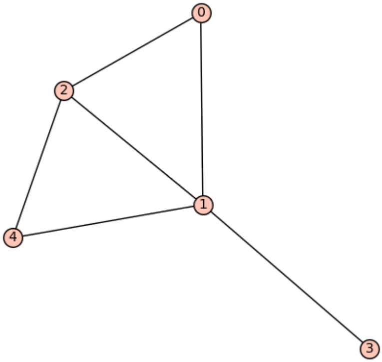
$$L(G) = D(G) - A(G).$$

L-eigenvalues: $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$.

Algebraic connectivity of G: $a(G) = \mu_2(G)$.

Example

Grafo G



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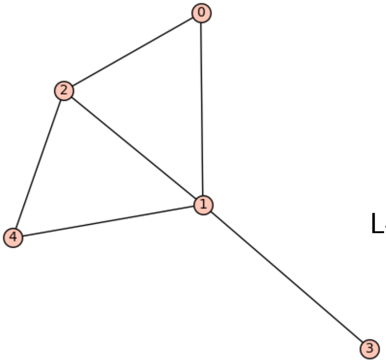
Grafo G



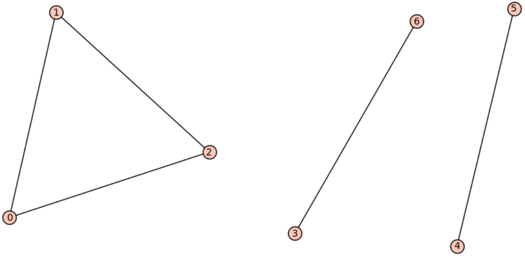
$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix}$$

L-eigenvalues: $\{5, 4, 2, 1, 0\}$.

Eigenvalues and Cuts: some results

Theorem 1 [Fiedler, 1973]

Let G be a graph on n vertices. Then $a(G) = 0$ if and only if G is disconnected.



L-eigenvalues: $\{3, 3, 2, 2, 0, 0, 0\}$

Vertex connectivity of a graph

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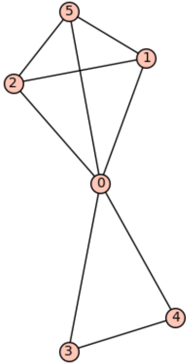


$$a(G) = k(G) = 1, k'(G) = 2$$

Fiedler (1973) bounded $a(G)$ by the edge and vertex connectivities.

Theorem 2 [Fiedler, 1973]

Let G be a non-complete graph on n vertices. Then

$$a(G) \leq k(G) \leq k'(G).$$

Next, we introduce the Max-Cut problem.

Max-cut problem

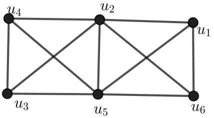
The maximum cut problem consists of finding a cut $\delta(S)$ in G for which

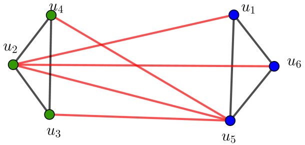
$$c(\delta(S)) = \sum_{(i,j)\in\delta(S)} w_{ij},$$

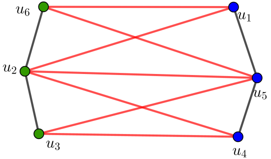
is **maximum**.

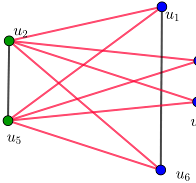
The maximum cut problem: an example

Consider the graph $G = (V, E)$ and let $w_{ij} = 1$ for all $(i, j) \in E$.









Mathematical formulation to the Max-Cut problem:

Maximize $\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - x_i x_j)$

subject to:

$$x_i \in \{-1, 1\} \text{ for all } i = 1, \ldots, n.$$

Note that...

if $x_i = x_j$, then $1 - x_i x_j = 0$; if $x_i \neq x_j$ then $1 - x_i x_j = 2$.

What a about the complexity?

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The max-cut problem is...

NP-Hard.

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However...

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The max-cut problem is...

NP-Hard.

However...

it is known to be polynomially solvable for some classes of graphs (for instance, planar graphs. See Hadlock, 1975).

F. Hadlock: Finding a maximum cut of a planar graph in polynomial time, SIAM J. on Comp. 4 (1975) 221-225.

The signless Laplacian of a graph

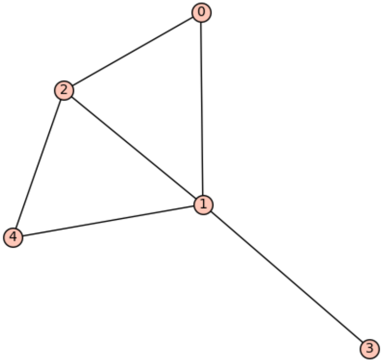
Let $A(G)$ be the (0,1)-adjacency matrix of a graph and $D(G)$ be the diagonal matrix where its entries are the degree of the vertices. The signless Laplacian of G is defined as follows:

$$Q(G) = D(G) + A(G).$$

Write $q_1(G) \geq \cdots \geq q_n(G) \geq 0$ for the eigenvalues of $Q(G)$.

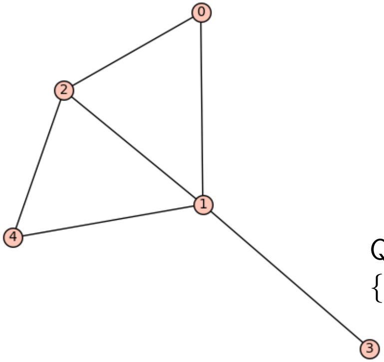
Example

Grafo G



Example

Grafo G



$$Q(G) = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Q-eigenvalues:
$\{5.78, 2.71, 2.00, 1.00, 0.51\}$.

Result: $q_n(G) = 0$ iff G is bipartite

The Max-Cut and the eigenvalues of A

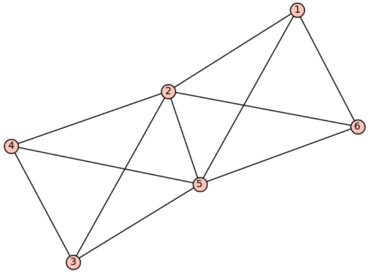
The smallest eigenvalue of $A(G)$ provides an upper bound to the Max-Cut.

Proposition 3 [Delorme and Poljak (1993) Let $G = (V, E)$ be a graph with n vertices, W be the sum of all edge weights of G and $\lambda_n(G)$ be the smallest eigenvalue of A. Then,

$$maxcut(G) \leq \frac{W}{2} - \frac{n\lambda_n(G)}{4}.$$

C. Delorme and S. Poljak, Laplacian eigenvalues and the maximum cut problem, Math. Programming 62 (1993), 557–574.

Let G be the graph below.



The A-eigenvalues of G are: $\{3.82, 1, -1, -1, -1, -1.82\}$

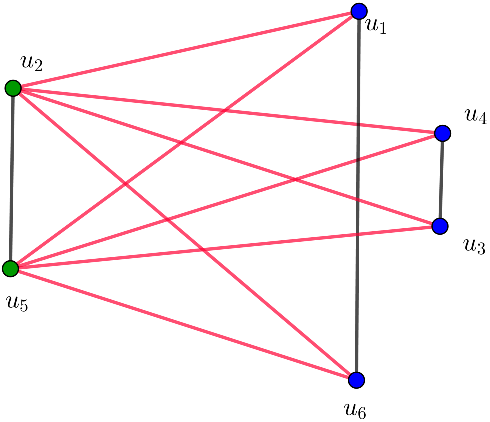
In this case, $\lambda_n(G) = -1.82, W = 11, n = 6$. Therefore,

$$maxcut(G) \leq \frac{W}{2} - \frac{n\lambda_n}{4} = 8.24$$

$$maxcut(G) \leq 8.24.$$

The Max-Cut and the eigenvalues of A

Notice that.... $maxcut(G) = 8$.



A generalization of the previous result.

Theorem 4 [Nikiforov, 2016] Let G be a graph of order n and size m, and let $mc_k(G)$ be the maximum size of a k-cut of G. It is shown that

$$mc_k(G) \leq \frac{k-1}{k}\left(m - \frac{n\lambda_n}{2}\right).$$

The Max-Cut and the eigenvalues of L

The largest eigenvalue of $L(G)$ provides an upper bound to the Max-Cut.

Proposition 4 [Mohar and Poljak, 1990] Let $G = (V, E)$ be a graph with n vertices, W be the sum of all edge weights of G and $\mu_1(G)$ be the largest eigenvalue of L. Then,

$$maxcut(G) \leq \frac{n\, \mu_1(G)}{4}.$$

B. Mohar, S. Poljak, Eigenvalues and the max-cut problem, Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 2, 343-352.

The Max-Cut and the eigenvalues of Q

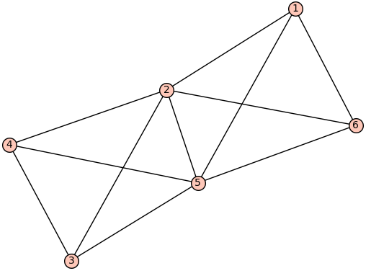
The smallest eigenvalue of $Q(G)$ provides an upper bound to the Max-Cut.

Proposition 5 [de Lima et al., 2011] Let $G = (V, E)$ be a graph with n vertices, m edges and $q_n(G)$ be the smallest eigenvalue of Q. Then,

$$maxcut(G) \leq m - \frac{nq_n(G)}{4}.$$

L. de Lima, C. Oliveira, N. Abreu, V. Nikiforov, The smallest eigenvalue of the signless Laplacian, Linear Algebra and its Applications, 435 (10) 2011, pp 2570-2584, 2011.

Let G be the graph below.



The Q-eigenvalues of G are: $\{8, 4, 4, 2, 2, 2.\}$

In this case, $q_n(G) = 2, W = 11, n = 6$. Therefore,

$$maxcut(G) \leq W - \frac{nq_n}{4} = 11 - \frac{6 \times 2}{4} = 8$$

$$maxcut(G) \leq 8.$$

Homework

Problems: Find all graphs such that:

(1) $maxcut(G) = n\mu_1/4$ (Exact graphs)

(2) $maxcut(G) = m - \frac{nq_n}{4}$ (Q-exact graphs)

(3) $maxcut(G) = m/2 - \frac{n\lambda_n}{4}$. (A-exact graphs)

Some extremal graphs were presented by...

B. Mohar, S. Poljak, Eigenvalues and the max-cut problem, Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 2, 343-352.

L. de Lima, J. Alencar, On graphs with adjacency and signless Laplacian matrix eigenvectors entries in $\{-1, +1\}$*, Linear Algebra and its applications, 614C (2021) pp. 301-315.*

Thank you!

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