Eigenvalues of Graphs: Some Tools and Problems

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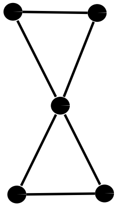
Graphs, Adjacency Matrices and Eigenvalues



- G **graph** on n vertices
- The **adjacency matrix** A is an $n \times n$ matrix where $A(x, y)$ equals the number of edges between x and y.
- The **eigenvalues** of A: $\lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{1-\sqrt{17}}{2},\ -1,\ -1,\ 1,\ \frac{1+\sqrt{17}}{2}$$

Some tools - Walks

Proposition

Let G be a graph with adjacency matrix A. For any ℓ and any vertices x and y, $A^{\ell}(x,y)$ equals the number of walks of length ℓ that start at x and end at y.

The complete bipartite graph $K_{a,b}$

$$A = \begin{bmatrix} 0 & J_{a,b} \\ J_{b,a} & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(A) = 2 \Rightarrow 0 \text{ eig. w. mult.} a+b-2$$

$$A^2 = \begin{bmatrix} bJ_a & 0 \\ 0 & aJ_b \end{bmatrix} \Rightarrow \lambda_1^2 + \lambda_{a+b}^2 = 2ab, \lambda_1 + \lambda_{a+b} = 0$$

$$\lambda_1 = \sqrt{ab}, \lambda_{a+b} = -\sqrt{ab}.$$

Some tools - Equitable Partitions

Definition

Let G be a graph with vertex set V. A partition $V = X_1 \cup \cdots \cup X_r$ is called **equitable** if there exist numbers $b_{j,\ell}, 1 \leq j, \ell \leq r$ such that for any $j, \ell \in \{1, \ldots, r\}$ and any vertex $x \in X_j$, the number of neighbors of x in X_ℓ equals $b_{j,\ell}$.

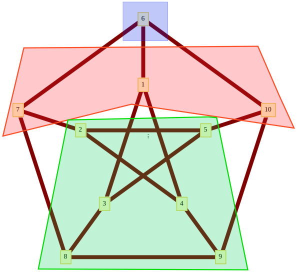


Figure: Equitable Partion

Some tools - Equitable Partitions

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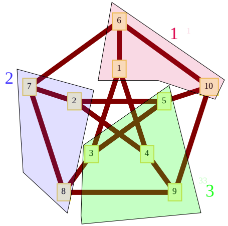

Figure: Not Equitable Partition

Some tools - Equitable Partitions

Definition

Let G be a graph with vertex set V. A partition $V = X_1 \cup \cdots \cup X_r$ is called equitable if there exist numbers $b_{j,\ell}, 1 \le j, \ell \le r$ such that for any $j, \ell \in \{1, \ldots, r\}$ and any vertex $x \in X_j$, the number of neighbors of x in X_ℓ equals $b_{j,\ell}$. The quotient matrix of the equitable partition is $B = (b_{j,\ell})_{1 \le j,\ell \le r}$.

Proposition

If G has an equitable partition, then the eigenvalues of the quotient matrix are eigenvalues of the adjacency matrix of G.

$K_{a,b}$ has equitable partition with quotient matrix $\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$ whose eigenvalues are $\sqrt{ab}, -\sqrt{ab}$.

Some tools - Equitable Partitions

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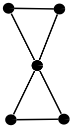


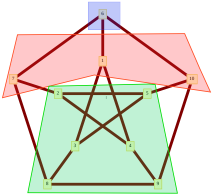
There is an equitable partition with quotient matrix $\begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$ (eigs $\frac{1\pm\sqrt{17}}{2}$).

Trabalho de casa Find an equitable partition with quotient matrix $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and its eigenvalues. Generalize to k triangles sharing a vertex.

Some tools - Equitable Partitions

Proposition

If G has an equitable partition, then the eigenvalues of the quotient matrix are eigenvalues of the adjacency matrix of G.



The Petersen graph has an equitable partition with quotient matrix $\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ whose eigenvalues are $3, 1, -2$.

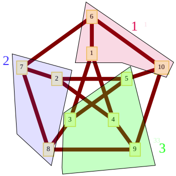
The eigenvalues of the Petersen graph are $3^{(1)}, 1^{(5)}, -2^{(4)}$.

Some tools - Eigenvalue Interlacing

Proposition

If G has a ~~an equitable~~ partition, then the eigenvalues of the quotient matrix **interlace** *eigenvalues of the adjacency matrix of G:*

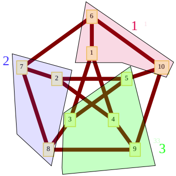
$$\lambda_{j+n-r}(A) \le \lambda_j(B) \le \lambda_j(A), 1 \le j \le r.$$



The quotient matrix entries are **the average degrees** between or inside the parts.

Some tools - Eigenvalue Interlacing

Petersen: $-2 \leq -2 \leq -2 \leq -2 < 1 \leq 1 \leq 1 \leq 1 \leq 1 \leq 1 < 3$



Trabalho de casa

The quotient matrix entries are the average degrees between or inside the parts. $B = \begin{bmatrix} 4/3 & 1/3 & 4/3 \\ 1/3 & 4/3 & 4/3 \\ 1 & 1 & 1 \end{bmatrix}$. Eigenvalues ?

Spectral Graph Theory

Eigenvalues of Regular Graphs

1. If G is a connected d-regular graph with n vertices, then

$$-d \leq \lambda_n \leq \cdots \leq \lambda_2 < \lambda_1 = d$$

2. If G is d-regular, then G is bipartite iff $\lambda_n = -d$.

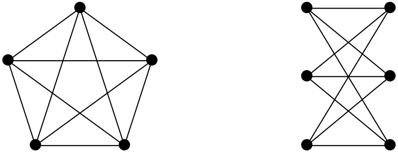


Figure: $K_5 : 4^{(1)}, -1^{(4)}$; $K_{3,3} : 3^{(1)}, 0^{(4)}, -3^{(1)}$

Large graphs with given valency and diameter

Theorem (Moore bound)

If G is a d-regular graph with diameter 2, it has at most $d^2 + 1$ vertices.

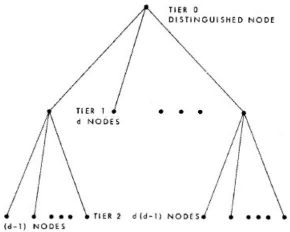


Figure: The Moore bound $d^2 + 1$

Problem

What are the d-regular graphs with diameter 2 and $d^2 + 1$ vertices?

Large graphs with given valency and diameter

Theorem (Hoffman-Singleton 1960)

If G is a d-regular graph with diameter 2 and d^2+1 vertices, then $d \in \{2,3,7,57\}$.

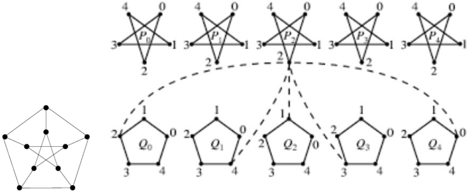


Figure: The Petersen graph and the Hoffman-Singleton graph

Bannai-Ito 1973; Damerell 1973: No Moore graphs with diameter$\geq$ 3.

Spectral characterization of graphs

Cospectral graphs

Two graphs G and H are **cospectral** (mates) if they have the same eigenvalues and are not isomorphic.

DS graphs

A graph G is **determined by its spectrum (DS)** if it has no cospectral mates.

The complete graphs, cycles, paths are DS.

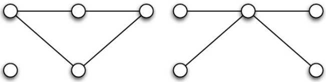


Figure: Smallest pair of cospectral graphs

Spectral characterization of graphs

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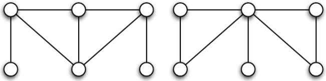
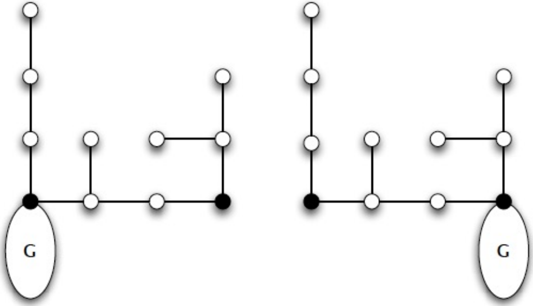


Figure: Smallest pair of connected cospectral graphs

Spectral characterization of graphs



Schwenk 1973, Godsil and McKay 1976

Almost all trees are **not DS** (**not DGS**).

Godsil-McKay

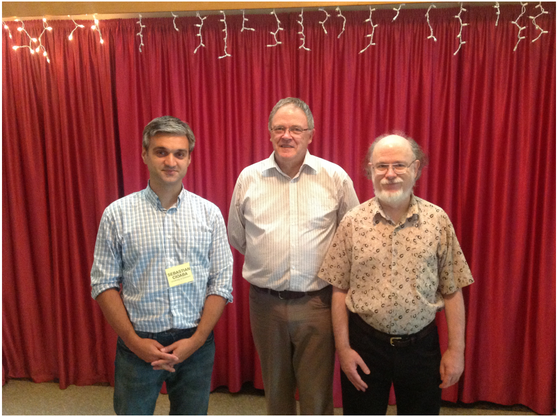


Figure: With Chris Godsil and Brendan McKay at Godsil65 Fest 2014

Godsil-McKay switching

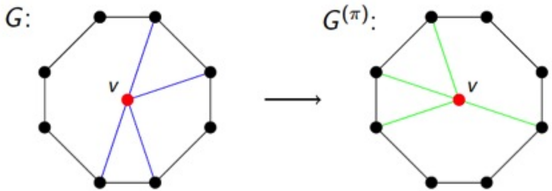
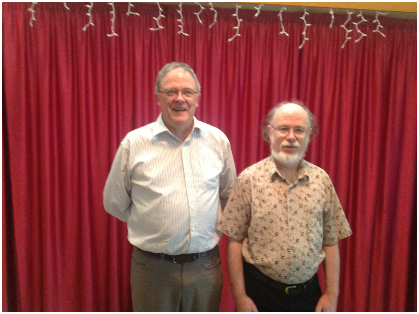


Figure: Godsil-McKay switching

Godsil-McKay 1982

Combinatorial method to construct cospectral graphs with cospectral complements.

Godsil-McKay switching



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Godsil-McKay switching

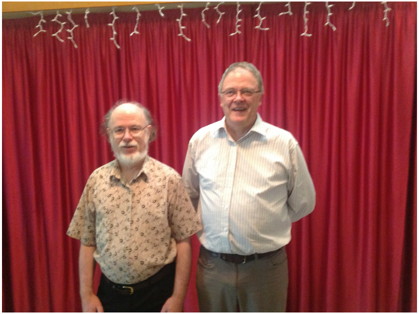


Figure: Godsil-McKay switching

Spectral characterization of graphs

van Dam-Haemers 2003

The DS status of many families of graphs (Johnson, Kneser) is not known.

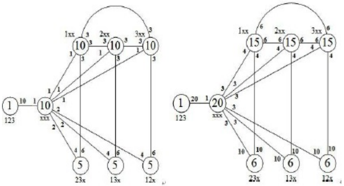


Figure: Kneser $K(8,3)$ not DS; $K(9,3)$???

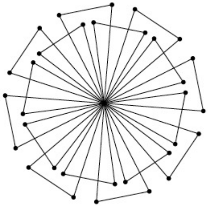
Haemers-Ramezani 2009: The Kneser graph $K(3k-1,k)$ not DS.

Cioabă-Haemers-Johnston-McGinnis 2018: Union of some graphs in the Johnson scheme not DS.

Spectral characterization of graphs

Conjecture (Wang, J.F., Belardo, F., Huang, Q.X., Borovicanin, B. 2010)

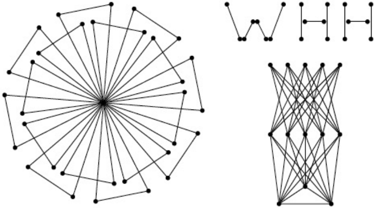
The Friendship graph F_k is DS for every $k \geq 1$.



Spectral characterization of graphs

Theorem (Cioabă, Haemers, Vermette and Wong 2015)

The Friendship graph F_k is DS for every $k \geq 1$ except for $k = 16$.



Spectral characterization of graphs

Conjecture (Haemers 2010)

Almost all graphs are DS.

Sensitivity Conjecture

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Any induced subgraph on $2^{n-1}+1$ vertices of the n-dimensional cube contains a vertex of degree at least $\sqrt{n}$.

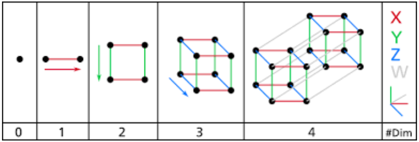

Figure: The n-dim cubes for $0 \leq n \leq 4$

Chung, Füredi, Graham, Seymour 1988

$\sqrt{n}$ is sharp; proved lower bound of $\frac{\log_2 n}{2}$.

Hao Huang, Proof of Sensitivity Conjecture

Sensitivity Conjecture

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